# MATH 53, Solutions to Practice for Midterm 2

Please email me if you spot any mistakes.

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**Problem 1.** Explain why $f(x, y) = 2x^2 + y^2$ must have absolute extrema constrained to the circle $x^2 + y^2 = 1$, and find them with Lagrange multipliers.

**Solution.** The circle is closed and bounded and the function $f$ is continuous, hence the Extreme Value Theorem guarantees the existence of absolute extrema of $f$ on $x^2 + y^2 = 1$.

The Lagrange multiplier system is

$$\begin{aligned} 4x &= \lambda 2x \\ 2y &= \lambda 2y \\ x^2 + y^2 &= 1. \end{aligned}$$

Remember, $\lambda$ is only an auxiliary variable, so we should try to get rid of it ASAP. If we multiply the first equation by $y$ and the second equation by $x$, we see that $4xy = \lambda 2xy = 2xy$. But this means $xy = 0$, so either $x = 0$ or $y = 0$. Hence our candidates are $(0, 1)$, $(0, -1)$ in the former case and $(1, 0)$, $(-1, 0)$ in the latter. Evaluation of $f$ at all four points shows that the maximum value of $f$ is 2, attained at the points $(1, 0)$, $(-1, 0)$, and the minimum value of $f$ is 1, attained at the points $(0, 1)$, $(0, -1)$.

By the way, you could've started out by noticing $f(x, y) = 2x^2 + y^2 = x^2 + 1$ on the constraint region! So the problem is equivalent to finding the extrema of $x^2+1$ on the same circle, which would result in simpler computations.

**Problem 2.** Evaluate the integral

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} \frac{y}{1+x^2}\, \mathrm{d}x\, \mathrm{d}y.$$

**Solution via switching order of integration.** Looking at the given integrand, I'd rather integrate with respect to $y$ than $x$. The region of integration is

$$\begin{aligned} y^2 &\le x \le 2 \\ 0 &\le y \le \sqrt{2}. \end{aligned}$$

To switch this to the integration order $\mathrm{d}y\,\mathrm{d}x$, you should draw a picture! Do that **now** before reading on.

The inequalities involving $y$ are $y^2 \le x$ (so $y \le \sqrt{x}$) and $0 \le y \le \sqrt{2}$. Since $x \le 2$, the inequality $y \le \sqrt{2}$ is redundant because we already have $y \le \sqrt{x}$. Hence

$$0 \le y \le \sqrt{x}.$$

Now we are left with the inequalities

$$\begin{aligned} 0 &\le \sqrt{x} \\ x &\le 2 \end{aligned}$$

where the first is always true–provided it exists! That is, it's a reminder to us that $x \ge 0$. Hence the $x$ bounds are just

$$0 \le x \le 2.$$

So our rewritten double integral is

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, \mathrm{d}y\, \mathrm{d}x &= \int_0^2 \frac{x}{2(1+x^2)}\, \mathrm{d}x \\ &= \frac{1}{4}\ln(1+x^2)\Big|_{x=0}^{2} = \boxed{\frac{1}{4}\ln 5}. \end{aligned}$$

**Solution via direct integration.** In this problem, it is actually possible to integrate the expression as written–provided you remember how to work with arctan (it can be integrated by parts).

$$\begin{aligned} \int_0^{\sqrt{2}} \int_{y^2}^{2} \frac{y}{1+x^2}\, \mathrm{d}x\, \mathrm{d}y &= \int_0^{\sqrt{2}} (y \arctan 2 - y\arctan(y^2))\, \mathrm{d}y \\ &= \left(\frac{1}{2} y^2 \arctan 2\right)\Big|_{y=0}^{\sqrt{2}} - \int_0^2 \frac{1}{2}\arctan(u)\, \mathrm{d}u \qquad \text{where } u = y^2 \\ &= \arctan 2 - \left(\left(\frac{u}{2}\arctan u\right)\Big|_{u=0}^{2} - \int_0^2 \frac{u}{2}\frac{1}{u^2+1}\, \mathrm{d}u\right) = \int_0^2 \frac{u}{2}\frac{1}{u^2+1}\, \mathrm{d}u \\ &= \frac{1}{4}\ln(1+x^2)\Big|_{x=0}^{2} = \boxed{\frac{1}{4}\ln 5}. \end{aligned}$$

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**Problem 3.** Find the volume of the region above the cone $z = \sqrt{x^2+y^2} - 2$ and the below the paraboloid $z = 4 - x^2 - y^2$.

**Solution.** Call the region $E$. You could set this up either as a triple integral or a double integral. As a triple integral it'd be

$$\iiint_E 1 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_R \int_{\sqrt{x^2+y^2}-2}^{4-x^2-y^2} 1 \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

where $R$ is some region in the $xy$-plane to be determined (namely, the projection of $E$ into the $xy$-plane). If we set it up as a double integral it'd be

$$\iint_R \left((4-x^2-y^2) - (\sqrt{x^2+y^2}-2)\right) \mathrm{d}x\,\mathrm{d}y$$

which is of course the same thing. Since the region $E$ is described by the inequality

$$\sqrt{x^2+y^2} - 2 \le z \le 4 - x^2 - y^2$$

the region $R$ is described by the inequality

$$\sqrt{x^2+y^2} - 2 \le 4 - x^2 - y^2$$

We're seeing an awful lot of $x^2+y^2$, so that's a hint to switch to polar. Our region $R$ is then described by $r-2 \le 4-r^2$, which after simplification (keeping in mind that we assume $r \ge 0$ as usual) we get $0 \le r \le 2$. Switching to polar our double integral becomes

$$\int_0^{2\pi} \int_0^2 (6 - r^2 - r) r \,\mathrm{d}r\,\mathrm{d}\theta = \boxed{\frac{32\pi}{3}}.$$

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**Problem 4.** Convert the following triple integral to Cartesian coordinates, but do not evaluate it.

$$\int_0^{\pi/2}\int_0^1\int_{r^2}^{r} r^2\cos\theta \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

**Solution.** We have

$$r^2\cos\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and the region of integration is

$$\begin{gathered} r^2 \le z \le r, \\ 0 \le r \le 1, \\ 0 \le \theta \le \pi/2. \end{gathered}$$

In Cartesian,

$$\begin{gathered} x^2 + y^2 \le z \le \sqrt{x^2+y^2}, \\ 0 \le \sqrt{x^2+y^2} \le 1, \\ x \ge 0,\ \ y \ge 0. \end{gathered}$$

The last inequality is a litle tricky to convert algebraically, because $\theta$ is not a well-defined function of $x, y, z$. But geometrically, it is easy to see the $\theta$ bounds just mean that $x, y \ge 0$. The region is bounded between an elliptic paraboloid and a cone.

Let's set this up with integration order $\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$. The innermost $z$-bounds are evident from the first inequality. Now we project onto the $xy$-plane and are left with the inequalities

$$\begin{gathered} x^2 + y^2 \le \sqrt{x^2+y^2}, \\ 0 \le \sqrt{x^2+y^2} \le 1, \\ x \ge 0,\ \ y \ge 0. \end{gathered}$$

but the first inequality is already implied by the second line. We have just a quarter-disk in the first quadrant. The middle $y$-bounds are thus $0 \le y \le \sqrt{1-x^2}$, and the outermost $x$-bounds are 0 to 1.

$$\boxed{\int_0^1\int_0^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}$$

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**Problem 5** (Stewart Ch. 15 Review #55)**.** Evaluate

$$\iint_R \frac{x-y}{x+y}\,\mathrm{d}x\,\mathrm{d}y$$

where $R$ is the square with vertices $(0,2),(1,1),(2,2),(1,3)$.

**Solution.** After drawing the region and inspecting the integrand, applying a change of variables seems like a good idea. Indeed, the region is described algebraically as

$$2 \le x+y \le 4,$$
$$-2 \le x-y \le 0.$$

If we also glance at the integrand, we see there's a lot of hints pointing to the change of variables $u = x-y, v = x+y$.

The region and function of integration are easy to rewrite in terms of $u, v$, so we need only figure out what happens to $\mathrm{d}x\,\mathrm{d}y$. The direct approach is to solve for $x, y$ in terms of $u, v$, which is really the "correct" way to describe the change of variables:

$$x = \frac{u+v}{2}$$
$$y = \frac{v-u}{2}$$
$$\left|\det\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\right| = 1/2$$

so $\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v$. Alternatively, one could compute the Jacobian of the *inverse* transformation $u = x-y, v = x+y$ and then take its reciprocal:

$$\left|\det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right| = 2$$

so $\mathrm{d}u\,\mathrm{d}v = 2\,\mathrm{d}x\,\mathrm{d}y$. Note however, that solving for $x, y$ in terms of $u, v$ may be necessary to rewrite the function of integration. After all this it's just direct computation:

$$\int_2^4 \int_{-2}^0 \frac{u}{v}\frac{1}{2}\,\mathrm{d}u\,\mathrm{d}v = \boxed{-\ln 2}.$$

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**Problem 6** (Stewart §16.2.2)**.** Let $C$ be the part of the curve $x^4 = y^3$ between the points $(1,1)$ and $(8,16)$. Compute $\int_C (x/y)\,\mathrm{d}s$.

**Solution.** First we parametrize $C$; there are many options but I will use

$$x = t^3,\ \ y = t^4,\ 1 \le t \le 2.$$

(Before reading on, check that this does in fact parametrize $C$.) Since

$$\mathrm{d}s = \sqrt{(3t^2)^2 + (4t^3)^2}\,\mathrm{d}t = t^2\sqrt{9+16t^2}\,\mathrm{d}t$$

our integral is

$$\begin{aligned}
\int_C (x/y)\,\mathrm{d}s &= \int_1^2 \frac{t^3}{t^4} t^2\sqrt{9+16t^2}\,\mathrm{d}t \\
&= \int_1^2 t\sqrt{9+16t^2}\,\mathrm{d}t \\
&= \frac{1}{32}\int_{25}^{73} \sqrt{u}\,\mathrm{d}u && \text{where } u = 9+16t^2 \\
&= \boxed{\frac{1}{48}(73^{3/2} - 125)}.
\end{aligned}$$

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**Problem 7.**

a) Find a function $f(x,y,z)$ such that

$$\nabla f = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle.$$

**Solution.** From $f_x(x,y,z) = 3x^2yz - 3y$, integration gives

$$f(x,y,z) = x^3yz - 3xy + C(y,z)$$

where $C(y,z)$ is some function of $y,z$ to be determined. Differentiation with respect to $y$ yields

$$f_y(x,y,z) = x^3z - 3x + C_y(y,z) = x^3z - 3x$$

thus $C_y(y,z) = 0$, meaning $C(y,z) = D(z)$, a function depending only on $z$. Computing $f_z$ gives

$$f_z(x,y,z) = x^3y + D'(z) = x^3y + 2z$$

therefore $D(z) = z^2 + E$ where $E$ can be any constant—and for convenience we take $E = 0$. Thus the following function will suffice.

$$\boxed{f(x,y,z) = x^3yz - 3xy + z^2}$$

b) Let $C$ be the line segment starting at $(0,0,2)$ and ending at $(0,3,0)$. Evaluate

$$\int_C \langle 3x^2yz - 3y + \arctan(yz), x^3z - 3x + 2\cos(x^2), x^3y + 2z\rangle \cdot d\mathbf{r}.$$

**Solution.** Notice that the vector field we are asked to integrate is

$$\nabla f + \langle \arctan(yz), 2\cos(x^2), 0\rangle.$$

From FTLI we get

$$\int_C (\nabla f)\cdot d\mathbf{r} = f(0,3,0) - f(0,0,2) = -4.$$

The other integral simplifies nicely once we notice that $dx = 0$ and $x = 0$ along $C$, so

$$\int_C \langle \arctan(yz), 2\cos(x^2), 0\rangle \cdot \langle dx, dy, dz\rangle = \int_C 2\cos 0\, dy = \int_C 2\, dy = 6.$$

If you do not understand how this simplification worked, you ought to directly parametrize $C$ and compute the integral that way (you will witness the simplifications happen naturally if you do that). Our final answer is $-4 + 6 = \boxed{2}$.

**Problem 8.** Let $C$ be the ellipse $x^2/4 + y^2 = 1$, oriented counterclockwise. Briefly explain why Green's theorem **cannot** be directly applied to the integral

$$\int_C \left(y \log_4(x^2+4y^2) + 3x^2y^2\cos(x^3)\right) \mathrm{d}x + \left(-7x + 2y\sin(x^3)\right) \mathrm{d}y.$$

Find a way around this issue and evaluate the integral.

**Solution.** If we wish to apply Green's theorem

$$\int_{\partial D} \langle P, Q\rangle \cdot \mathrm{d}\mathbf{r} = \iint_D (Q_x - P_y)\, \mathrm{d}x\, \mathrm{d}y$$

then it is not enough that $P, Q$ are nice along the boundary $\partial D$... we need $Q_x, P_y$ to be nice on the enclosed region $D$ as well! The glaring issue with the given integral is that $\log_4(x^2+4y^2)$ is not defined when $(x, y) = (0, 0)$, which is a point enclosed by $C$.

An easy way around this is to note that the given integral takes place on $C$, and all points on $C$ satisfy the equation $x^2/4 + y^2 = 1$ (by definition). Hence we may replace $\log_4(x^2+4y^2)$ by $\log_4 4 = 1$ without affecting the value of the given line integral. After we do that, we are in the clear to use Green's theorem. As $C$ is already oriented positively (i.e. counterclockwise),

$$\int_C \left(y + 3x^2y^2\cos(x^3)\right) \mathrm{d}x + \left(-7x + 2y\sin(x^3)\right) \mathrm{d}y = \iint_D \left((-7 + 6x^2y\cos(x^3)) - (1 + 6x^2y\cos(x^3))\right) \mathrm{d}x\, \mathrm{d}y$$

where $D$ is the region enclosed by $C$. This simplifies to

$$\iint_D -8\, \mathrm{d}x\, \mathrm{d}y = -8(2\pi) = \boxed{-16\pi}$$

because the area of the ellipse $D$ is $2\pi$.

Suppose you didn't know this last fact. You could derive it by either

- applying the change of variables $x = 2u$, $y = v$ to transform $D$ into a circular disk $S$ in the $uv$-plane, obtaining

$$\iint_D -8\, \mathrm{d}x\, \mathrm{d}y = \iint_S -16\, \mathrm{d}u\, \mathrm{d}v = -16\pi,$$

  or

- applying Green's theorem *again*, returning to an integral along $C$, but choosing a nicer vector field $\langle P, Q\rangle$ with $Q_x - P_y = -8$. For example,

$$\iint_D -8\, \mathrm{d}x\, \mathrm{d}y = \int_C \langle 8y, 0\rangle \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} -16(\sin t)^2\, \mathrm{d}t = -16\pi$$

  where we used the parametrization $x = 2\cos t$, $y = \sin t$. Note that the area formula $\int_C y\, \mathrm{d}x$ from Chapter 10 is a special case of this method.

**Problem 9.** Suppose we have an object of mass $M$ occupying some region $D$, and let $CM = (\bar{x}, \bar{y})$ be its center of mass. If $P = (a, b)$ is any point in the plane, we can define the moment of inertia about $P$ as

$$I_P = \int_D (\text{distance from } P \text{ to } (x, y))^2 \, \mathrm{d}m.$$

a) Calculate $I_P - I_{CM}$, leaving your final answer in terms of the constants $a, b, \bar{x}, \bar{y}, M$ only (there should be no integrals in your final answer).

**Solution.** We write everything out and cancel some terms:

$$\begin{aligned} I_P - I_{CM} &= \int_D \Big( \big((a-x)^2 + (b-y)^2\big) - \big((\bar{x}-x)^2 + (\bar{y}-y)^2\big) \Big) \, \mathrm{d}m \\ &= \int_D \Big( a^2 - 2ax + b^2 - 2by - \bar{x}^2 + 2\bar{x}x - \bar{y}^2 + 2\bar{y}y \Big) \, \mathrm{d}m. \end{aligned}$$

Observe that $\int_D \mathrm{d}m = M$, and $\bar{x}M = \int_D x \, \mathrm{d}m$ (similar equation for $\bar{y}$). Keep in mind also that everything other than $x, y$ is a constant and can be moved out of the integral.

$$I_P - I_{CM} = (a^2 + b^2 - \bar{x}^2 - \bar{y}^2)M + (2\bar{x} - 2a)\bar{x}M + (2\bar{y} - 2b)\bar{y}M$$

This already meets the requirements of the problem, but if you simplify further, you get

$$I_P - I_{CM} = \boxed{M\big((a - \bar{x})^2 + (b - \bar{y})^2\big)}$$

where the parenthesized part is the distance from $P$ to $CM$, squared.

b) Suppose that that our object is a circular cable of radius 2 centered at the origin with constant density and total mass $M = \pi$. Let $P = (1, 0)$. Find $I_P$. (Even if you did not solve (a), you can do this directly.)

**Solution using (a).** Let $C$ denote the circle. In this case, $CM = (0, 0)$ by symmetry and $I_{CM} = \int_C 4 \, \mathrm{d}m = 4M = 4\pi$. So the formula from (a) yields

$$I_P - 4\pi = \pi$$

hence $I_P = \boxed{5\pi}$.

**Solution by direct computation.** The length of the cable $C$ is $4\pi$, so the density is $M/(4\pi) = 1/4$.

$$\begin{aligned} I_P = \int_C \big((x-1)^2 + y^2\big) \, \mathrm{d}m &= \int_C \big((x-1)^2 + y^2\big) \frac{1}{4} \, \mathrm{d}s \\ &= \int_0^{2\pi} (5 - 4\cos t) \frac{1}{4} 2 \, \mathrm{d}t \\ &= \boxed{5\pi} \end{aligned}$$

where we used the parametrization $x = 2\cos t$, $y = 2\sin t$.